Higher-Order Fourier Analysis: Applications to Algebraic Property Testing

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Decision Problems

- Function $f : \mathbb{F}_2^n \to \{0, 1\}$.
- Function property $\mathcal{P}$.
  (such as linearity: $f(x) + f(y) \equiv f(x + y) \bmod 2$ for all x, y.)

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Q. Can we do something in sublinear or even in constant time?

Property testing

Definition

$f : \{0,1\}^n \to \{0,1\}$ is *ϵ-far* from $\mathcal{P}$ if,

$$d_{\mathcal{P}}(f) := \min_{g \in \mathcal{P}} \frac{\#\{x \in \{0,1\}^n \mid f(x) \neq g(x)\}}{2^n} \geq \epsilon.$$

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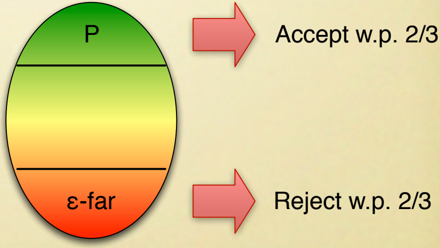
A *tester* for a property $\mathcal{P}$:
Given

- $f : \{0,1\}^n \to \{0,1\}$ as a query access.
- proximity parameter $\epsilon > 0$.



Testing $f \equiv 1$

Input: a function $f : \{0,1\}^n \to \{0,1\}$ and $\epsilon > 0$.
Goal: $f(x) = 1$ for every $x \in \{0,1\}^n$?

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2: Sample $x \in \{0,1\}^n$ uniformly at random.
3: **if** $f(x) = 0$ **then** reject.
4: Accept.

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- *Query complexity is $O(1/\epsilon) \Rightarrow$ constant!*

Backgrounds

Property testing was introduced by [RS96] for program checking.

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Q. Why do we study property testing?
A. Interested in

- ultra-efficient algorithms.
- connections to inapproximability, locally testable codes, and learning.
- the relation between local view and global property.

Local testability of affine-Invariant properties

Definition

$\mathcal{P}$ is *affine-invariant* if a function $f : \mathbb{F}_2^n \to \{0,1\}$ satisfies $\mathcal{P}$, then $f \circ A$ satisfies $\mathcal{P}$ for any bijective affine transformation $A : \mathbb{F}_2^n \to \mathbb{F}_2^n$.

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Definition

$\mathcal{P}$ is *(locally) testable* if there is a tester for $\mathcal{P}$ with $q(\epsilon)$ queries.

Local testability of affine-Invariant properties

Some specific locally testable affine-invariant properties:

- Degree-d polynomials [AKK+05, BKS+10]
- Fourier sparsity [GOS+11]
- Odd-cycle-freeness: There exist no $x_1, \ldots, x_{2k+1} \in \mathbb{F}_2^n$ such that $f(x_1) = \cdots = f(x_{2k+1}) = 1$ and $x_1 + \cdots + x_{2k+1} \equiv 0$ [BGRS12].

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In this talk, we review how we have attacked this question.

- One-sided error testable $\approx$ Affine-subspace hereditary
- Testable $\Leftrightarrow$ Estimable
- Two-sided error testable $\Leftrightarrow$ Regular-reducible

Higher order Fourier analysis has played a crucial role!

Fourier analysis

For $S \subseteq [n]$, define $\chi_S : \mathbb{F}_2^n \to \{-1,1\}$ as $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$.
$\{\chi_S\}$ forms an orthonormal basis for functions: $\mathbb{F}_2^n \to \mathbb{R}$:

$$\mathop{\mathbf{E}}_x[\chi_S(x)\chi_T(x)] = \mathop{\mathbf{E}}_x[(-1)^{\sum_{i \in S \triangle T} x_i}] = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases}$$

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$\Rightarrow$ A function $f : \mathbb{F}_2^n \to \mathbb{R}$ can be uniquely decomposed as

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S)\chi_S(x),$$

where $\widehat{f}(S) = \mathbf{E}_x[f(x)\chi_S(x)]$ measures the correlation of f with χ_S.
(Fourier coefficients)

Linearity testing

Input: a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ and $\epsilon > 0$.
Goal: $f(x) + f(y) \equiv f(x + y) \pmod 2$ for every $x, y \in \mathbb{F}_2^n$?

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Theorem ([BLR93])

- *If f is linear, always accepts. (one-sided error)*
- *If f is ϵ-far, rejects with probability at least $2/3$.*
- *Query complexity is $O(1/\epsilon)$*

Observation

For any $S \subseteq [n]$, we have

$$\epsilon \leq d_{\mathrm{LIN}}(f) = \Pr[f(x) \not\equiv \sum_{i \in S} x_i] = \Pr[(-1)^{f(x)} \neq \chi_S(x)]$$
$$= \mathbf{E}\Big[\frac{1-(-1)^{f(x)}\chi_S(x)}{2}\Big] = \frac{1-\widehat{g}(S)}{2} \quad (g := (-1)^f).$$

This fact can be used to analyze the rejection probability.

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Fourier analysis is

- powerful enough to study specific properties.
- not powerful enough to obtain general results.

Higher order Fourier analysis

We look at correlations with polynomials instead of linear functions.

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Main technical tools:

- *Decomposition theorem*
  A function can be decomposed into a structured part + pseudorandom part (with respect to low-degree polynomials)
- *Equidistribution theorem*
  "generic" polynomials look independently distributed.

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  "generic" polynomials look independently distributed.

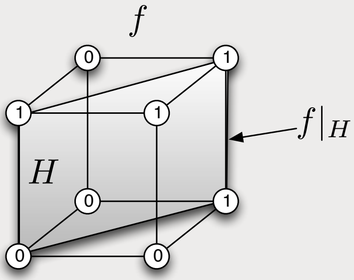
Caveat: In this talk, we do not touch most of technical foundations such as Gowers norm, rank, and bias.

Oblivious tester

Definition

An *oblivious tester* works as follows:

- Take a restriction $f|_H$.
  - H: random affine subspace of dimension $h(\epsilon)$.
- Output based only on $f|_H$.



Motivation: avoid "unnatural" properties such as $f \in \mathcal{P} \Leftrightarrow n$ is even.
For natural properties, $\exists$ a tester $\Rightarrow$ $\exists$ an oblivious tester [BGS10].

Decomposition theorem

$\mu_{f,h}$: the distribution of $f|_H$.

Observation

A tester cannot distinguish f from g if $\mu_{f,h} \approx \mu_{g,h}$.

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Theorem (Decomposition theorem)

Any function can be decomposed as $f = f_1 + f_2 + f_3$ for $d = d(\epsilon, h)$:

- *$f_1 = \Gamma(P_1, \ldots, P_C)$ for "generic" degree-d polynomials $\{P_i\}$.*
- *f_2: small L_2 norm.*
- *f_3: uncorrelated with degree-d polynomials.*

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- f_2*: small* L_2 *norm.*
- f_3*: uncorrelated with degree-d polynomials.*

The *pseudorandom parts* f_2 and f_3 do not affect $\mu_{f,h}$ much.
$\Rightarrow$ we can focus on the *structured part* f_1.

Decomposition theorem for Fourier analysis

Any function $f : \mathbb{F}_2 \to \{-1,1\}$ can be decomposed as:

$$f = \sum_{S \subseteq [n] : |\widehat{f}(S)| > \epsilon} \widehat{f}(S)\chi_S(x) + \sum_{S \subseteq [n] : |\widehat{f}(S)| \leq \epsilon} \widehat{f}(S)\chi_S(x).$$

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- By Perseval's identity $\sum_S \widehat{f}(S)^2 = 1$
  $\Rightarrow$ the former depends only on $O(1/\epsilon^2)$ many linear functions.
- The latter has negligible Fourier coefficients.

One-sided error testable $\approx$ Affine-subspace hereditary

Affine-subspace hereditary

Definition

A property $\mathcal{P}$ is *affine-subspace hereditary* if $f \in \mathcal{P} \Rightarrow f|_H \in \mathcal{P}$ for any affine subspace H.

Ex.:

- degree-d polynomials, Fourier sparsity, odd-cycle-freeness
- $f = gh$ for some polynomials g, h of degree $\leq d - 1$.
- $f = g^2$ for some polynomial g of degree $\leq d - 1$.

Characterization of one-sided error testability

Conjecture ([BGS10])

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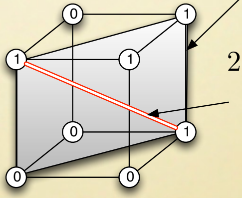
Proof sketch:

1. Suppose $f \in \mathcal{P}$ and $f|_H \notin \mathcal{P}$



2. $\exists f|_K$, rejected by the tester

3. f is also rejected w.p.> 0, contradiction.

Alternative formulation via linear forms

Think of *affine-triangle-freeness*:
No $x, y_1, y_2 \in \mathbb{F}_2^n$ s.t. $f(x+y_1) = f(x+y_2) = f(x+y_1+y_2) = 1$.

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$\Leftrightarrow$ No $x, y_1, y_2 \in \mathbb{F}_2^n$ s.t.

$$f(L_1(x, y_1, y_2)) = \sigma_1 \text{ for } L_1(x, y_1, y_2) = x + y_1 \text{ and } \sigma_1 = 1,$$
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We call this *$(A = (L_1, L_2, L_3), \sigma = (\sigma_1, \sigma_2, \sigma_3))$-freeness*.

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We call this *$(A = (L_1, L_2, L_3), \sigma = (\sigma_1, \sigma_2, \sigma_3))$-freeness*.

- A is called an *affine* system of *linear forms*.
  $\Rightarrow$ well studied in higher order Fourier analysis.

Testability of subspace hereditary properties

Observation

The following are equivalent:

- $\mathcal{P}$ *is affine-subspace hereditary.*
- *There exists a (possibly infinite) collection* $\{(A^1, \sigma^1), \ldots\}$ *s.t.* $f \in \mathcal{P} \Leftrightarrow f$ *is* (A^i, σ^i)*-free for each* i.

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Theorem ([BFH+13])

If each (A^i, σ^i) *has bounded complexity, then the property is testable with one-sided error.*

Proof idea

Let's focus on the case $f = \Gamma(P_1, \ldots, P_C)$ and $\mathcal{P}$ = affine $\triangle$-freeness.

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$$\Pr_{x,y_1,y_2}[f(x+y_1) = f(x+y_2) = f(x+y_1+y_2) = 1]$$
$$\geq \Pr_{x,y_1,y_2}[P_i(L_j(x,y_1,y_2)) = P_i(L_j(x^*,y_1^*,y_2^*))\ \forall i \in [C], j \in [3]],$$

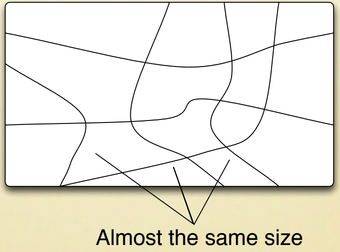
which is non-negligibly high from the *equidistribution theorem*.
$\Rightarrow$ Random sampling works.

Equidistribution theorem

The space $\mathbb{F}_2^n$ can be divided according to $\{P_i(L_j(x))\}_{i\in[C],j\in[3]}$.

Theorem (Equidistribution theorem)

If each P_i is "generic" enough, then each cell has almost the same size.

$\mathbb{F}_2^n =$ 

Equidistribution theorem for Fourier analysis

Let $P_1, \ldots, P_C : \mathbb{F}_2^n \to \mathbb{F}_2$ be linear functions. If $P_i(x)$ is of the form $\sum_{j \in S_i} x_j$, and $\mathbf{1}_{S_1}, \ldots, \mathbf{1}_{S_C}$ are linearly independent, then for any $\sigma_1, \ldots, \sigma_C$,

$$\Pr_x[P_i(x) = \sigma_i \ \forall i \in [C]] = \frac{\#\{x \in \mathbb{F}_2^n \mid \sum_{j \in S_i} x_j = \sigma_i \ \forall i \in [C]\}}{2^n}$$
$$= 2^{-|C|}.$$

$\Rightarrow$ Equidistributed.

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Theorem ([HL13])

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Algorithm:

1: $H \leftarrow$ a random affine subspace of a constant dimension.

2: **return** Output $d_{\mathcal{P}}(f|_H)$.

Intuition behind the proof

Why can we hope $d_{\mathcal{P}}(f) \approx d_{\mathcal{P}}(f|_H)$*?*

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(Oversimplified argument)

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- If $f = \Gamma(P_1, \ldots, P_C)$, then $\mu_{f,h}$ is determined by Γ and degrees of $P_1, \ldots, P_C$ (rather than *P_i's themselves*).
- $f = \Gamma(P_1, \ldots, P_C)$ and $f_H = \Gamma(P_1|_H, \ldots, P_C|_H)$ share the same Γ and degrees.

  $\Rightarrow \mu_{f,h} \approx \mu_{f|_H,h}$.

  $\Rightarrow d_{\mathcal{P}}(f) \approx d_{\mathcal{P}}(f|_H)$.

Two-sided error testability $\Leftrightarrow$ Regular-reducibility

Structured part

Recall that, for $f = \Gamma(P_1, \ldots, P_C) + f_2 + f_3$,

$\mu_{f,h}$ is determined by Γ and degrees of P_i's.

Let's use them as a (constant-size) sketch of f.

Regularity-instance (simplified)

Definition

A *regularity-instance* I is a tuple of

- a structure function $\Gamma : \mathbb{F}_2^C \to [0,1]$,
- a complexity parameter $C \in \mathbb{N}$,
- a degree-bound parameter $d \in \mathbb{N}$,
- a degree parameter $\mathbf{d} = (d_1, \ldots, d_C) \in \mathbb{N}^C$ with $d_i < d$,

Satisfying a regularity-instance

Definition

Let $I = (\gamma, \Gamma, C, d, \mathbf{d})$ be a regularity-instance.
f *satisfies* I if it is of the form

$$f(x) = \Gamma(P_1(x), \ldots, P_C(x)) + \Upsilon(x),$$

where

- P_i is a polynomial of degree d_i,
- $P_1, \ldots, P_C$ are "generic" enough.
- Υ is uncorrelated with degree-$(d-1)$ polynomials.

Testing regularity-instances

Theorem ([Yos14a])

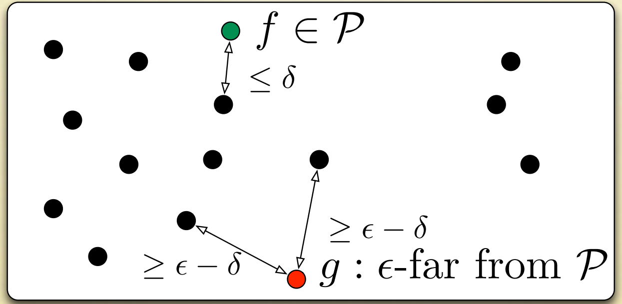
For any regularity-instance I, there is a tester for the property of satisfying I.

Algorithm:

1: $H \leftarrow$ a random affine subspace of a constant dimension.
2: **if** $f|_H$ is close to satisfying I **then** accept.
3: **else** reject.

Regular-reducibility

A property $\mathcal{P}$ is *regular-reducible* if for any $\delta > 0$, there exists a set $\mathcal{R}$ of constant number of regularity-instances such that:



Characterization of two-sided error testability

Theorem

An affine-invariant property $\mathcal{P}$ is testable

$$\Updownarrow$$

$\mathcal{P}$ is regular-reducible.

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Proof sketch:

- Regular-reducible $\Rightarrow$ testable
  Regularity-instances are testable, and testability implies estimability [HL13]. Hence, we can estimate the distance to $\mathcal{R}$.
- Testable $\Rightarrow$ regular-reducible
  The behavior of a tester depends only on $\mu_{f,h}$. Since Γ and $\mathbf{d}$ determines the distribution, we can find $\mathcal{R}$ using the tester.

Notes

- We need to deal with "non-classical" polynomials instead of polynomials.
- Another characterization of testability was shown by introducing "functions limits" [Yos14b].
- Applications of the characterizations:
  - Low-degree polynomials.
  - Having a low spectral norm $\sum_S |\widehat{f}(S)|$.

Summary

Higher order Fourier analysis is useful for studying property testing as

- we care about the distribution $\mu_{f,h}$ for $h = O(1)$,
- which is determined by the structured part given by the decomposition theorem.

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Higher order Fourier analysis is useful for studying property testing as

- we care about the distribution $\mu_{f,h}$ for $h = O(1)$,
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We are almost done, *qualitatively*.

- one-sided error testability $\approx$ affine-subspace hereditary (of bounded complexity)
- two-sided error testability $\Leftrightarrow$ regular-reducibility.

Future direction

Property Testing

- Other groups:
  - Abelian $\Rightarrow$ higher order Fourier analysis exists [Sze12].
  - Non-Abelian $\Rightarrow$ representation theory? [OY16]
- Why is affine invariance easier to deal with than permutation invariance?

Other applications of higher order Fourier analysis.

- Coding theory [BG16, BL15a].
- Learning theory [BHT15].
- Complexity theory [BL15b].
- Algorithms for polynomials [Bha14].